

for $s = -1, 0, 1, 2, 3, \dots$, where P_{-1} will give the velocity-potential of the initial relative motion.

It is now an easy matter to write down the equations of motion of the body, if the motion is known to be of the type we are considering.

As has already been mentioned, it is proposed in a future paper to give an account of the small motion of a sphere in a form which is numerically intelligible. The expansions used in this particular case have been proved to converge for all finite values of t .

A Theory of Colour Vision.

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At the present time there is no generally accepted theory of colour vision, and there is considerable divergence of opinion as to the tests which should be applied for colour blindness. The Young-Helmholtz theory certainly explains the facts of colour mixing, but it is doubtful whether we can deduce from these facts the existence of three primary colour sensations.

For, if we represent colours by particles on a plane, making the masses of the particles proportional to the luminosities of the colours, and represent mixtures of these colours by the rule for determining the centre of gravity of the particles, as is done on Newton's colour diagram, then, if any triangle be drawn enclosing all the possible points, its corners may be taken as representing the three primary colours, and these corners may take up an infinite number of positions. The usual procedure is to make two sides of this triangle fit the curve representing the spectrum as closely as possible, and take two corners of the triangle on the ends of the curve. Two of the primary colours are then definite colours of the spectrum, while the third is a sensation which can never be produced pure. But there is a certain amount of arbitrariness about this proceeding.

Also there is no doubt that the belief in primary colours is partly due to the experiments in mixing pigments which we are all familiar with from childhood, and rests rather on an inveterate tradition than on reason. Just as there were four elements, fire, water, earth and air, so there ought to be primary colours. I have several times demonstrated to schoolboys, who

knew nothing about colour vision, that red, yellow, and blue were not primary colours by mixing the colours with a rotating disc, and have noted in each case that the result of the experiment was to convince them, not that there were no primary colours, but that red, yellow, and blue were not the right ones.

I have therefore thought it worth while, in this short paper, to try to explain the facts of colour mixing by a theory which does not depend on primary colour sensations. This theory may be regarded as a mathematical formulation of Dr. Edridge-Green's views, though I do not know he would accept it as such ; I was not aware of his views until my own conclusions were almost definitely formed. The paper is divided into two parts: the first, which deals with the retinal process, applies ideas already more or less familiar ; but the second part, which deals with the cerebral process, uses an idea quite original in its application to colour vision.

(1) The curve represented by the full line in fig. 1 is taken from a paper

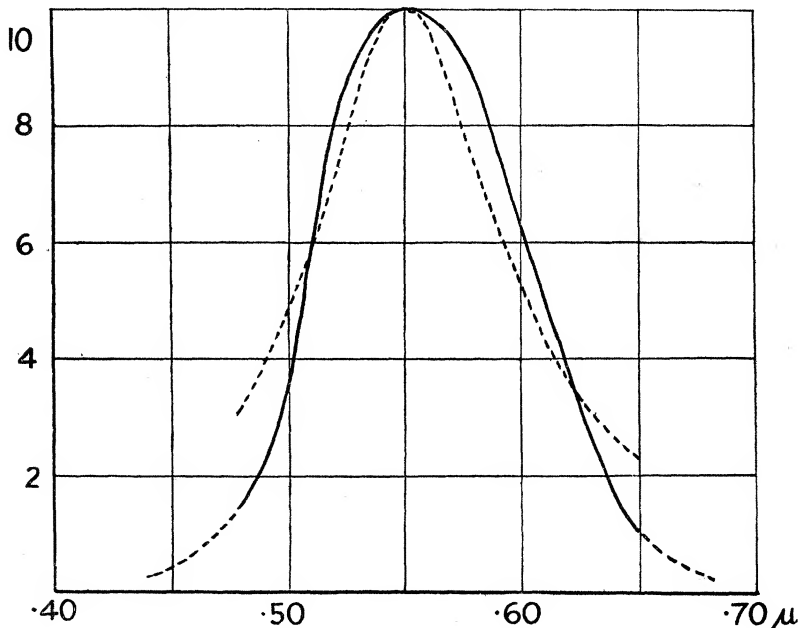


FIG. 1.

by H. E. Ives* on the photometry of light of different colours. It represents the sensitiveness of the eye for light of different wave-lengths. We learn from it, for example, that if the same quantities of energy are in succession

* 'Phil. Mag.,' [6], vol. 24, p. 860 (1912). The curve has recently been verified and extended by P. G. Nutting, 'Phil. Mag.,' February, 1915, p. 301.

changed into light of the wave-lengths 6400 Å.U., 6000 Å.U., and 5600 Å.U., the luminosity of the sensation produced will be in the ratio of the numbers 1·7, 6·4, and 9·8. The curve gives the mean of the results for 18 observers, and the field of observation had the brightness of a magnesium oxide surface undergoing an illumination of 25 metre-candles and viewed by an eye in which the pupil is at its normal aperture. The dotted part of the curve was extrapolated from the observations.

It is obvious from the curve that the eye has a maximum of sensitiveness in the green, and that the sensitiveness falls away rapidly on both sides of this, towards the red and violet. What is this due to? The most obvious explanation is to suppose that there exists in the eye a very great number of vibrators, with a free period in the green, and that these execute forced vibrations under the influence of the light wave. The amplitude of the forced vibrations is a maximum when the free period of the vibrators coincides with the period of the incident light.

We may represent the motion of one of these vibrators by the equation

$$\frac{d^2x}{dt^2} + h \frac{dx}{dt} + n^2x = E \cos \omega t, \quad (1)$$

where x represents the displacement of a typical vibrator from its position of equilibrium and $E \cos \omega t$ is the force per unit mass exerted on it by the incident light wave. The rate at which energy is being absorbed by the vibrator is given by

$$E \frac{dx}{dt} \cos \omega t. \quad (2)$$

The solution of (1) consists of two parts, the free vibration and the forced vibration. If the vibrator is left to itself, the free vibrations die down, but whenever it receives an impact, they are renewed again. In (2) it is necessary to substitute only the part of dx/dt due to the forced vibration, because, when the other part is multiplied by $\cos \omega t$, the average value of the product is zero. When the part of dx/dt due to the forced vibration is calculated and substituted, (2) becomes

$$-\frac{E^2\omega}{\{(n^2-\omega^2)^2+h^2\omega^2\}^{\frac{1}{2}}} \cos \omega t \sin \left(\omega t - \tan^{-1} \frac{h\omega}{n^2-\omega^2} \right).$$

The mean value of this is

$$\frac{E^2h\omega^2}{2\{(n^2-\omega^2)^2+h^2\omega^2\}}.$$

When $2\pi c/\lambda$ is substituted for ω and we write $a^2 = (2\pi c/n)^2$ and $b^2 = (2\pi ch/n^2)^2$, this becomes

$$\frac{E^2h(2\pi c)^2}{2n^4} \frac{\lambda^2}{(\lambda^2 - a^2)^2 + b^2\lambda^2}.$$

The intensity of the incident light is proportional to E^2 . If we assume that the luminosity is proportional to the energy absorbed, and omit a constant factor, the ratio of absorbed to incident energy or, in other words, the visibility of the radiation, is proportional simply to

$$\frac{\lambda^2}{(\lambda^2 - a^2)^2 + b^2 \lambda^2}. \quad (3)$$

This expression is represented by the dotted curve in fig. 1 for $a = 0.55 \mu$ and $b = 0.10 \mu$. For different values of a and b the agreement between the observed and calculated curves might be better, but not much better. As a whole we may say that (3) represents the visibility curve roughly.

Let us now consider what happens to (3) when E^2 is increased. It remains constant. That is, the sensation is proportional to the intensity of the light. This is opposed to experience. According to Fechner's law, if dI is the smallest increase in intensity which produces a perceptible increase in luminosity,

$$dI/I \text{ is constant.}$$

It seems natural therefore to assume that the increase in luminosity is proportional to dI/I , or in other words, if E^2 suffers a small increase, the increase in energy absorbed is proportional, not to $d(E^2)$ as given by (3), but to $d(E^2)/E^2$. This assumption implies that some of the vibrators go out of action when E^2 is increased. When the energy of the vibrator reaches a critical value, the force attaching the vibrator to its centre snaps, the latter then ceases to absorb light energy, and a chemical change takes place. This critical value is not the same for all the vibrators, but varies from vibrator to vibrator. We may identify the chemical change with the bleaching of the visual purple, but this identification is not necessary to the explanation.

Let N be the number of vibrators, then, per unit area of retina, when the amplitude of the incident light wave is E , and let us assume that E is increased. The increase in energy absorbed is proportional to $d(NE^2)$. By our assumption

$$d(NE^2) = c \frac{d(E^2)}{E^2},$$

where c is a constant. This gives

$$N = \frac{1}{E^2} (N_0 E_0^2 + c \log E^2 / E_0^2),$$

where N_0 is the number of vibrators acting when the amplitude of the wave is E_0 . The expression shows, as was intended, that N diminishes as E increases, but is not suitable for small values of E . However, Fechner's law itself does not hold for very small values of the intensity.

Of course we are not to suppose that if E is constant the same identical vibrators remain in action all the time, but there are two processes going on in opposite directions which balance one another, visual purple being bleached and constantly restored. When E increases, the point of equilibrium is shifted. Owing to the bleaching and restoration of the visual purple we must suppose the vibrators to be in a perpetual state of agitation. Their free vibrations are constantly being renewed.

If the intensity of illumination is reduced, it is found that the visibility curve undergoes a change. Its maximum is gradually shifted towards the green, reaching the limiting position of 0.50μ when the intensity is very small, the curve becoming at the same time narrower near the maximum. The phenomenon is known as the Purkinje effect, and can be shown quite easily with a small spectroscope. If a bright continuous spectrum is observed, and the width of the slit is gradually decreased, the maximum of brightness shifts from the yellow, and when the spectrum has become very faint it has reached the green. According to von Kries, the effect can be explained by assuming that the rods in the retina are chiefly responsible for vision at low intensities and the cones for vision at high intensities. I do not, however, think it necessary to assume two different mechanisms. We can explain the effect with one system of vibrators in either of two ways.

First we may suppose the vibrators embedded in a medium with a yellow colour, something like potassium chromate. At low intensities the energy is absorbed near the surface of the medium; at high intensities most of the vibrators near the surface will have become bleached. Consequently a larger proportion of the energy is absorbed at a greater depth, but the energy on the blue side of the maximum suffers a greater absorption by the medium on the way in, and there is not so much left for the vibrators to absorb at this depth. Thus the maximum of absorption is shifted towards the yellow. I do not know the exact colour of the yellow spot in the eye and am unable to say whether it would produce the required effect.

The alternative explanation is to assume that all the vibrators have not the same period, but that the value of a varies from vibrator to vibrator, $a = 0.55 \mu$ being only a mean value. Then, if those that decompose more easily have a smaller value of a , the shift is explained. This explanation seems to me the better one. The assumption of different values of a follows naturally from the assumption that the vibrators decompose at different intensities. Also, Ive's visibility curve (the full curve in fig. 1) is very like a probability curve, *i.e.* it looks as if it could be represented by the expression

$$e^{-k(\lambda - 0.55)^2}, \quad (4)$$

where k is a constant. If the different values of a are distributed around the mean value according to the law of error, the visibility curve will be the sum of a great number of small curves of the type represented by (3). If these component curves have a small value of b , their individual peculiarities will not appear in the resultant, but only the law according to which they are grouped about their mean, and hence we shall obtain an expression similar to (4) for the resultant. By proceeding in this way we can obtain a much better agreement between theory and experiment than in fig. 1.

The comparison of theory with experiment might be carried further, but I do not think it wise to press the analogy with the pendulum too closely in the case of a living substance. It is sufficient that the reception of light by the eye can be represented broadly by one class of vibrators in the manner shown. Also, it should be noted that there is no decided evidence in favour of three classes of vibrators; *i.e.* the visibility curve has not three separate maxima. It has a simple form, probably simpler than the three component visibility curves into which it would have to be resolved to meet the views of those who believe in the existence of three independent primary sensations with an independent mechanism for each.

(2) So much for the retinal part of the theory. We suppose the vibrators to set up waves in the nerves and that the nerves carry these waves to the brain. The analogue of the telephone may be useful in this respect; the vibrators may be likened to the diaphragm and the nerve to the telephone wire. There is, however, one important difference. When a musical note is sung into the telephone it is transmitted correctly along the wire and reproduced at the other end, because the diaphragm reproduces accurately the sound wave. It is fundamental to the theory given here that the vibrator does not reproduce the light wave accurately, owing to its being subject to too many disturbing influences. No matter how monochromatic the incident wave is, the wave transmitted along the nerve is not monochromatic. It is as if the diaphragm in the transmitter were subject to disturbing influences of such a nature that the person at the other end hears a medley of musical notes over a range of half an octave on each side of the note originally sung into the transmitter.

We may approach the subject in another way. The vibrator may start from rest, in which case we have the free vibration and forced vibration superimposed. The free vibration may die down, then be renewed again by an impact; the period of the vibrator may change slightly. The vibrator may then decompose and, after a short rest, reunite and start again. Now an irregular motion of this sort may be regarded as due to a superposition of

sine waves, *i.e.* the displacement of the vibrator may be expressed as a Fourier integral.

Lord Rayleigh has shown in a classic paper* that if the displacement of the vibrator is given as a function of the time by $f(t)$, and if k be written for $2\pi/\lambda$, where λ is the wave-length of the Fourier component, then the distribution of the energy over the different components is given by

$$\frac{1}{\pi} \int_0^\infty \left[\left(\int_{-\infty}^{+\infty} \cos kv f(v) dv \right)^2 + \left(\int_{-\infty}^{+\infty} \sin kv f(v) dv \right)^2 \right] dk.$$

Unfortunately there is no suitable expression for $f(t)$; consequently we cannot evaluate the distribution mathematically. But from first principles it is possible to get an idea of the result in particular cases.

Let us suppose, for example, that yellow light of wave-length 0.59μ is falling on the retina, then the distribution of the energy over the different waves set up in the nerve may be represented by some such curve as is

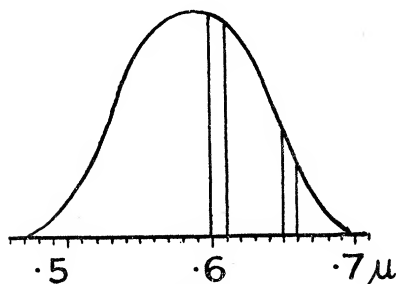


FIG. 2.

shown in fig. 2. In the curve the abscissa is λ , as this is more useful for practical purposes than k . The whole area of the curve represents the energy received by the brain, *i.e.*, the luminosity of the sensation. If we draw any two ordinates, the area enclosed between them represents the energy of the range of wave-lengths between the points at the feet of the ordinates. Thus we

see from the curve, that between 0.60 and 0.61μ there is enclosed more than twice as much energy as between 0.65 and 0.66μ . The maximum of the curve will coincide with the wave-length of the incident light, but the curve will in general not be symmetrical; more of its area will be on the same side of the maximum as the free vibrations are. If the vibrator executed a pure forced vibration for a very long time, the curve would be infinitely narrow. Thus the breadth of the curve is a measure of the degree of disturbance, to which the vibrator is subject.

I regard the energy curve as shown in fig. 2 as characteristic for the light impression on the brain. We are accustomed to associate three qualities with a light impression, namely luminosity, hue, and saturation. The area of the curve gives the luminosity of the impression, the position of the maximum the hue, and the narrowness of the curve the degree of saturation.

It is easy with such curves to represent the phenomena of colour mixing.

* 'Phil. Mag., vol. 27, pp. 460-469 (1889).

For example, in fig. 3, I have drawn two energy curves of equal luminosity to represent lithium red and thallium green; their sum, which is shown by the

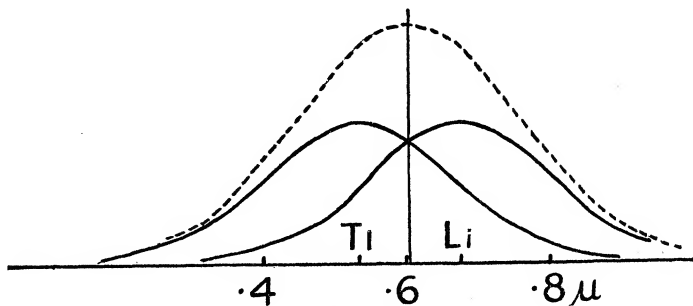


FIG. 3.

dotted line, is a curve of somewhat similar shape with its maximum at 0.603μ . In fig. 4, I have drawn the same two curves, but with the luminosity of the

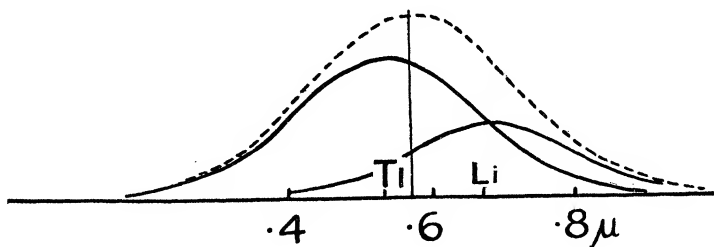


FIG. 4.

red equal only to half the luminosity of the green. The resultant curve, shown by the dotted line, has its maximum at 0.580μ . These maxima are not at the right positions, but the curves drawn are intended merely to be illustrative; I have given them the shape of probability curves. It will be noticed they extend beyond the limits of the visible spectrum. There is no theoretical objection to this.

But, it will be asked, how does this theory explain the apparent trichromatism of our ordinary sensations. We must here fall back on the reason given by Dr. Edridge-Green, namely that the colour-perceiving centre in the brain is not sufficiently developed to discriminate between the character of adjacent curves. Two curves must be widely different in shape and position, before the colour-perceiving centre can detect the difference. A curve has an infinite number of points on it. The colour-perceiving centre is so badly developed that, as far as it is concerned, the curve is sufficiently specified by three points on it, provided that these points are distributed over the spectrum. We can therefore represent our energy curve by three points.

Since the sensation of luminosity is better developed than that of colour, I have preferred to represent the curve by three rectangles as in the figs. 5, 6, and 7, which represent respectively a sodium yellow, a lithium green, and

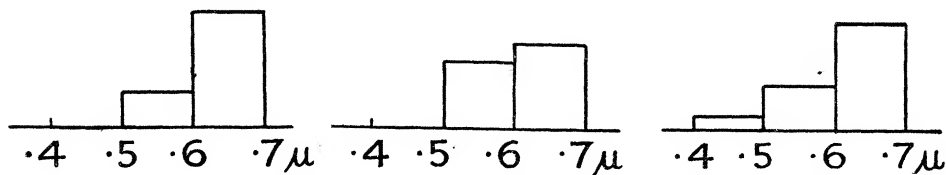


FIG. 5.

FIG. 6.

FIG. 7.

a white light, all of equal luminosity. The areas of the rectangles may be regarded as the amount of stimulation of the three primary sensations of the Young-Helmholtz theory; indeed, I took the value of the ordinates from one of Sir William Abney's curves, merely exaggerating the size of the third component in fig. 7 to make it visible. The diagrams may therefore be regarded as a means of representing the results of the Young-Helmholtz theory. I believe, however, that they are more than this, that they actually are energy curves—crude ones, it is true, but sufficiently representative for the discriminating power of the colour-perceiving centre. The sides of the rectangles in the diagrams have been placed at 0.4, 0.5, 0.6, and 0.7 μ , but this is not essential. The diagrams are, to some extent, a connecting link between the Young-Helmholtz theory and that of Dr. Edridge-Green.

The question arises as to the exact shape of the energy curve produced by monochromatic stimulation of the retina. I see no way of determining it experimentally. The colour of the sensation produced by monochromatic stimulation varies with the intensity of the latter, becoming whiter for high intensities. This fact supports the theory developed here. For, when the impressed force on the vibrator is increased, the latter is more strongly damped, and hence the energy of the Fourier components of its motion more widely distributed over the spectrum.